The Joint Spectral Radius
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The paper "A note on the joint spectral radius" gives a natural definition of the joint spectral radius of two matrices A and B:

$$\rho(A, B) = \liminf_{N \to \infty} \quad (\text{largest norm of any product with } N \text{ factors})^{1/N}.$$

The product can have A's and B's in any order (and the lim inf is actually a limit). I was very proud to be a joint author with Gian-Carlo, but I am now a little ashamed that we never thought seriously about how to compute this number ρ. For a single matrix it equals the largest magnitude of the eigenvalues $\lambda(A)$. But products of A and B can produce norms and eigenvalues that are very hard to estimate (as N increases) from the two matrices. The Lyapunov exponent is a similar number, using averages over products of length N instead of maxima, and it suffers from the same difficulty (impossibility?) in actual computation. An equivalent definition of ρ is the infimum over all matrix norms of $\max(\|A\|, \|B\|)$. The definitions extend directly to sets of more than two matrices, and an l_p norm joint spectral radius has also proved useful [57, 71].

Every few years, Gian-Carlo would ask me whether anyone ever read our paper. After I had tenure, I could tell him the truth: "not often". Part of the reason may have been the relatively unfamiliar journal, and the nonexistence of the Internet, and even the atypically obscure language that he had chosen to express our (very simple) idea. In recent years I could change my answer! The joint spectral radius suddenly found application in wavelet theory, especially in the work of Ingrid Daubechies and Jeff Lagarias [31]. Of special interest is the question whether $\rho < 1$, so that products with more and more factors of A and B all go to zero.

The application comes in solving the key equation in wavelet theory—the refinement equation (or dilation equation) for the scaling function. Its special feature is the presence of two time scales t and $2t$. Written as a system for a vector unknown as in [108], it involves two matrices $m(0)$ and $m(1)$:

$$\phi(t) = m(0)\phi(2t) + m(1)\phi(2t - 1) \quad \text{on} \quad (0, 1).$$

At the point t with binary expansion $.t_1 t_2 t_3 t_4 \ldots$, either $m(0)$ or $m(1)$ enters on the right side according to whether t_1 is 0 or 1. And the recursion continues, involving all the bits in the expansion of t:

$$\phi(t) = m(t_1)\phi(.t_2 t_3 t_4 \ldots) = m(t_1)m(t_2)\phi(.t_3 t_4 t_5 ...) = \ldots$$

A nearby point T begins with the same bits as t. To prove continuity of the solution is to show that $\phi(T)$ is close to $\phi(t)$. Both $m(0)$ and $m(1)$ have an eigenvalue at 1, but its effect cancels in the difference $\phi(T) - \phi(t)$. What matters is the restrictions A and B of $m(0)$ and $m(1)$ to their common invariant subspace, orthogonal to $(1, \ldots, 1)$. Continuity will hold if the product of A and B in every order is small—thus if $\rho(A, B) < 1$. The exact value of ρ determines the exponent of Hölder continuity of $\phi(t)$.

A beautiful result of Berger and Wang [4] allows the option of estimating ρ from the eigenvalues of the products, instead of their norms. The eigenvalues (we take their absolute values) approach from below and the norms from above. Thus

$$\rho(A,B) = \liminf_{N \to \infty} \quad (\text{largest } |\text{eigenvalue}| \text{ of any product with } N \text{ factors})^{1/N}.$$

Lagarias and Wang [71], and also Gurvits [?], asked whether this limit is necessarily attained for a finite product. This "finiteness conjecture" was disproved in 2000 by Bousch and Mairesse [9], using their deep results for dynamical systems. They gave the form of a 2 by 2 counterexample, but not explicit matrices A and B. Maesumi (private communication) conjectures that such counterexamples form a set of measure zero. Blondel, Theys, and Vladimirov [7] have just established that

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b & 0 \\ b & b \end{bmatrix}$$

yield counterexamples for infinitely many b. A spcific b remains unknown in November 2001.

The logical decidability of $\rho \leq 1$ has also been an active question. Blondel and Tsitsiklis [8] have now given a negative answer. Perhaps we will never have an explicit counterexample to finiteness? Gian-Carlo would have been quite happy, if he knew that our simple definition led to such profound questions (and answers).

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